

# Weighted Approximation theorem for Choldowsky generalization of the $q$ -Favard-Szász operators

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ABSTRACT. we study the convergence of these operators in a weighted space of functions on a positive semi-axis and estimate the approximation by using a new type of weighted modulus of continuity and error estimation.

Keywords:  $q$ -Favard- Szász operators, error estimation.

## 1. Introduction and auxiliary results

The classical Favard–Szász–perators are given as follows

$$(1.1) \quad S_n(f, ;) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right)$$

These operators and the generalizations have been studied by several other researcher ( see. [1]-[7]) and references there in. In 1969, Jakimovski and Leviatan [11] introduced the Favard-Szász type operator, by using Appell polynomials  $p_k(x)$  ( $k \geq 0$ ) defined by

$$g(u)e^{-ux} = \sum_{k=0}^{\infty} p_k(x)u^k,$$

where  $g(z) = \sum_{n=0}^{\infty} a_n z^n$  is analytic function in the disc  $|z| < R$ ,  $R > 1$  and  $g(1) \neq 0$ ,

$$P_{n,t}(f, x) = \frac{e^{nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right)$$

and they investigated some approximation properties of these operators.

Atakut et al.[10] defined a choldowsky type of Favard-Szász operators as follows:

$$(1.2) \quad P_n^*(f, x) = \frac{e^{\frac{nx}{b_n}}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{nx}{b_n}\right) f\left(\frac{k}{n}b_n\right),$$

with  $b_n$  a positive increasing sequence with the properties  $\lim_{n \rightarrow \infty} b_n = \infty$  and  $\lim_{n \rightarrow \infty} \frac{b_n}{n} = 0$ . They also studied some approximation properties of the operators.

Recently, A. Karaiza [12] defined Choldowsky type generalization of the Favard-Szász operators as follows:

$$(1.3) \quad P_n^*(f; q; x) = \frac{E_q^{\frac{[n]_q x}{b_n}}}{A(1)} \sum_{k=0}^{\infty} \frac{P_k(q; \frac{[n]_q x}{b_n})}{[k]_q!} f\left(\frac{[k]_q}{[n]_q} b_n\right),$$

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where  $\{P_k(q; \cdot)\} \geq 0$  is a  $q$ -Appell polynomial set which is generated by

$$A(u) \frac{e^{\frac{[n]_q x}{b_n}}}{u} = \sum_{k=0}^{\infty} \frac{P_k(q; \frac{[n]_q x}{b_n}) u^k}{[k]_q!}$$

and  $A(u)$  is defined by  $A(u) = \sum_{k=0}^{\infty} a_k u^k$ .

Motivated by these results, in this paper we study weighted approximation and error estimation of these operators. During the last two decades, the applications of  $q$ -calculus emerged as a new area in the field of approximation theory. The rapid development of  $q$ -calculus has led to the discovery of various generalizations of Bernstein polynomials involving  $q$ -integers. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design and solutions of differential equations. To make the article self-content, here we mention certain basic definitions of  $q$ -calculus, details can be found in [9] and the other recent articles. For each non negative integer  $n$ , the  $q$ -integer  $[n]_q$  and the  $q$ -factorial  $[n]_q!$  are, respectively, defined by

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1, \\ n, & q = 1, \end{cases}$$

and

$$[n]_q! = \begin{cases} [n]_q [n-1]_q [n-2]_q \dots [1]_q, & n = 1, 2, \dots, \\ 1, & n = 0. \end{cases}$$

Then for  $q > 0$  and integers  $n, k, k \geq n \geq 0$ , we have

$$[n+1]_q = 1 + q[n]_q \quad \text{and} \quad [n]_q + q^n [k-n]_q = [k]_q.$$

The  $q$ -derivative  $D_q f$  of a function  $f$  is defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, x \neq 0.$$

The  $q$ -analogues of the exponential function are given by

$$e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!},$$

and

$$E_q^x = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{x^n}{[n]_q!}.$$

The exponential functions have the following properties:

$$D_q(e_q^{ax}) = ae_q^{ax}, \quad D_q(E_q^{ax}) = aE_q^{ax}, \quad e_q^x E_q^{-x} = E_q^x e_q^{-x} = 1.$$

LEMMA 1. [12] *The following hold:*

- (i)  $P_n^*(e_0; q; x) = 1,$
- (ii)  $P_n^*(e_1; q; x) = x + \frac{D_q(A(1))E_q^{-\frac{[n]_q x}{b_n}} e_q^{\frac{[n]_q x}{b_n}}}{A(1)} \frac{b_n}{[n]_q},$
- (iii)  $P_n^*(e_2; q; x) = x^2 + \frac{E_q^{-\frac{[n]_q x}{b_n}} e_q^{\frac{[n]_q x}{b_n}} [qD_q(A(q)) + D_q(A(1))]}{A(1)} \frac{b_n x}{[n]_q} + \frac{D_q^2(A(1))E_q^{-\frac{[n]_q x}{b_n}} e_q^{\frac{[n]_q x}{b_n}}}{A(1)} \frac{b_n^2}{[n]_q^2}.$

where  $e_i(x) = x^i, i = 0, 1, 2.$

Now we give an auxiliary lemma for the Korovkin test functions.

- LEMMA 2. (i)  $P_n^*(t-x; q; x) = \frac{D_q(A(1))E_q^{-\frac{[n]_q x}{b_n}} e_q^{\frac{[n]_q x}{b_n}}}{A(1)} \frac{b_n}{[n]_q},$
- (iii)  $P_n^*((t-x)^2; q; x) = \frac{E_q^{-\frac{[n]_q x}{b_n}} e_q^{\frac{[n]_q x}{b_n}} [qD_q(A(q)) - D_q(A(1))]}{A(1)} \frac{b_n x}{[n]_q} + \frac{D_q^2(A(1))E_q^{-\frac{[n]_q x}{b_n}} e_q^{\frac{[n]_q x}{b_n}}}{A(1)} \frac{b_n^2}{[n]_q^2}.$

## 2. Weighted approximation

Let  $B_{x^2}[0, \infty)$  be the set of functions defined on  $[0, \infty)$  satisfying the condition  $|f(x)| \leq M_f(1+x^2)$ , where  $M_f$  is a constant depending on  $f$  only. By  $C_{x^2}[0, \infty)$ , we denote subspace of all continuous functions belonging to  $B_{x^2}[0, \infty)$ . Also, let  $C_{x^2}^*[0, \infty)$  be the subspace of all  $f \in C_{x^2}[0, \infty)$  for which  $\lim_{x \rightarrow \infty} \frac{f(x)}{1+x^2}$  is finite. The norm on  $C_{x^2}^*[0, \infty)$  if  $\|f\|_{x^2} = \sup_{x \in [0, \infty)} \frac{|f(x)|}{1+x^2}$ . For any positive number  $a$ , we define

$$\omega_a(f, \delta) = \sup_{|t-x| \leq \delta} \sup_{x, t \in [0, a]} |f(t) - f(x)|,$$

and denote the usual modulus of continuity of  $f$  on the closed interval  $[0, a]$ . We know that for a function  $f \in C_{x^2}[0, \infty)$ , the modulus of continuity  $\omega_a(f, \delta)$  tends to zero.

Now, we shall discuss the weighted approximation theorem, when the approximation formula holds true on the interval  $[0, \infty)$ .

**THEOREM 1.** *For each  $f \in C_{x^2}^*[0, \infty)$ , we have*

$$\lim_{n \rightarrow \infty} \|P_n^*(f; q; x) - f\|_{x^2} = 0.$$

**PROOF.** Using the theorem in [8] we see that it is sufficient to verify the following three conditions

$$(2.1) \quad \lim_{n \rightarrow \infty} \|P_n^*(t^r; q; x) - x^r\|_{x^2} = 0, \quad r = 0, 1, 2.$$

Since,  $P_n^*(1, x) = 1$ , the first condition of (2.1) is satisfied for  $r = 0$ . Now,

$$\begin{aligned} \|P_n^*(t; q; x) - x\|_{x^2} &= \sup_{x \in [0, \infty)} \frac{|P_n^*(t; q; x) - x|}{1+x^2} \\ &\leq \sup_{x \in [0, \infty)} \left| \left( x + \frac{D_q(A(1))E_q^{-\frac{[n]_q x}{b_n}} e_q^{\frac{[n]_q x}{b_n}}}{A(1)} \frac{b_n}{[n]_q} - x \right) \frac{1}{1+x^2} \right| \\ &\leq \sup_{x \in [0, \infty)} \left| \left( \frac{D_q(A(1))E_q^{-\frac{[n]_q x}{b_n}} e_q^{\frac{[n]_q x}{b_n}}}{A(1)} \frac{b_n}{[n]_q} \right) \frac{1}{1+x^2} \right| \end{aligned}$$

which implies that

$$\|P_n^*(t, x) - x\|_{x^2} = 0.$$

Finally,

$$\begin{aligned} \|P_n^*(t^2; q; x) - x^2\|_{x^2} &= \sup_{x \in [0, \infty)} \frac{|P_n^*(t^2; q; x) - x^2|}{1+x^2} \\ &\leq \sup_{x \in [0, \infty)} \left| x^2 + \frac{E_q^{-\frac{[n]_q x}{b_n}} e_q^{\frac{[n]_q x}{b_n}} [qD_q(A(q)) + D_q(A(1))] b_n x}{A(1)} \frac{1}{[n]_q} + \frac{D_q^2(A(1))E_q^{-\frac{[n]_q x}{b_n}} e_q^{\frac{[n]_q x}{b_n}}}{A(1)} \frac{b_n^2}{[n]_q^2} - x^2 \right| \frac{1}{1+x^2} \end{aligned}$$

which implies that  $\|P_n^*(t^2; q; x) - x^2\|_{x^2} \rightarrow 0$  as  $[n]_q \rightarrow \infty$ . Thus proof is completed.  $\square$

We give the following theorem to approximate all functions in  $C_{x^2}[0, \infty)$ .

**THEOREM 2.** *For each  $f \in C_{x^2}[0, \infty)$  and  $\alpha > 0$ , we have  $\lim_{[n]_q \rightarrow \infty} \sup_{x \in [0, \infty)} \frac{|P_n^*(f; q; x) - f(x)|}{(1+x^2)^{1+\alpha}} = 0$ .*

**PROOF.** For any fixed  $x_0 > 0$ ,

$$\begin{aligned} \sup_{x \in [0, \infty)} \frac{|P_n^*(f; q; x) - f(x)|}{(1+x^2)^{1+\alpha}} &\leq \sup_{x \leq x_0} \frac{|P_n^*(f, x) - f(x)|}{(1+x^2)^{1+\alpha}} + \sup_{x \geq x_0} \frac{|P_n^*(f; q; x) - f(x)|}{(1+x^2)^{1+\alpha}} \\ &\leq \|P_n^*(f; q; x) - f\|_{C[0, x_0]} + \|f\|_{x^2} \sup_{x \geq x_0} \frac{|P_n^*(1+t^2; q; x)|}{(1+x^2)^{1+\alpha}} + \sup_{x \geq x_0} \frac{|f(x)|}{(1+x^2)^{1+\alpha}}. \end{aligned}$$

The first term of the above inequality tends to zero from Theorem 3. By Lemma 2 for any fixed  $x_0 > 0$  it is easily seen that  $\sup_{x \geq x_0} \frac{|P_n^*(1+t^2; q; x)|}{(1+x^2)^{1+\alpha}}$  tends to zero as  $[n]_q \rightarrow \infty$ . We can choose  $x_0 > 0$  so large that the last part of the above inequality can be made small enough. Thus the proof is completed.  $\square$

### 3. Error Estimation

The usual modulus of continuity of  $f$  on the closed interval  $[0, b]$  is defined by

$$\omega_b(f, \delta) = \sup_{|t-x| \leq \delta, x, t \in [0, b]} |f(t) - f(x)|, \quad b > 0.$$

We first consider the Banach lattice, for a function  $f \in E$ ,  $\lim_{\delta \rightarrow 0^+} \omega_b(f, q; \delta) = 0$ , where

$$E := \left\{ f \in C[0, \infty) : \lim_{x \rightarrow \infty} \frac{f(x)}{1+x^2} \text{ is finite} \right\}.$$

The next theorem gives the rate of convergence of the operators  $P_n^*(f, x)$  to  $f(x)$ , for all  $f \in E$ .

**THEOREM 3.** *Let  $f \in E$  and  $\omega_{b+1}(f, q; \delta)$ ,  $0 < q < 1$  be its modulus of continuity on the finite interval  $[0, b+1] \subset [0, \infty)$ , where  $a > 0$  then, we have*

$$\|P_n^*(f; q; x) - f\|_{C[0, b]} \leq M_f(1+b^2)\delta_n(b) + 2\omega_{b+1}\left(f, \sqrt{\delta_n(b)}\right).$$

**PROOF.** The proof is based on the following inequality

$$(3.1) \quad \|P_n^*(f; q; x) - f\| \leq M_f(1+b^2)P_n^*((t-x)^2, x) + \left(1 + \frac{P_n^*(|t-x|, x)}{\delta}\right) \omega_{b+1}(f, \delta).$$

For all  $(x, t) \in [0, b] \times [0, \infty) := S$ . To prove (3.1), we write

$$S = S_1 \cup S_2 := \{(x, t) : 0 \leq x \leq b, 0 \leq t \leq b+1\} \cup \{(x, t) : 0 \leq x \leq b, t > b+1\}.$$

If  $(x, t) \in S_1$ , we can write

$$(3.2) \quad |f(t) - f(x)| \leq \omega_{b+1}(f, |t-x|) \leq \left(1 + \frac{|t-x|}{\delta}\right) \omega_{b+1}(f, \delta)$$

where  $\delta > 0$ . On the other hand, if  $(x, t) \in S_2$ , using the fact that  $t-x > 1$ , we have

$$(3.3) \quad \begin{aligned} |f(t) - f(x)| &\leq M_f(1+x^2+t^2) \\ &\leq M_f(1+3x^2+2(t-x)^2) \\ &\leq N_f(1+b^2)(t-x)^2 \end{aligned}$$

where  $N_f = 6M_f$ . Combining (3.2) and (3.3), we get (3.1). Now from (3.1) it follows that

$$\begin{aligned} |P_n^*(f; q; x) - f(x)| &\leq N_f(1+b^2)P_n^*((t-x)^2; q; x) + \left(1 + \frac{P_n^*(|t-x|, x)}{\delta}\right) \omega_{b+1}(f, \delta) \\ &\leq N_f(1+b^2)P_n^*((t-x)^2, x) + \left(1 + \frac{[P_n^*((t-x)^2, x)]^{1/2}}{\delta}\right) \omega_{b+1}(f, \delta). \end{aligned}$$

By Lemma 2 we have

$$\begin{aligned} P_n^*(t-x)^2 &\leq \delta_n(b). \\ \|P_n^*(f; q; x) - f\| &\leq N_f(1+b^2)\delta_n(b) + \left(1 + \frac{\sqrt{\delta_n(b)}}{\delta}\right) \omega_{b+1}(f, \delta). \end{aligned}$$

Choosing  $\delta = \sqrt{\delta_n(b)}$ , we get the desired estimation.  $\square$

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